Comparing Fractions Using Reasoning and Sense-Making

Facilitation Guide

INTRODUCTION

The purpose of this sequence of tasks is to help prospective elementary teachers develop fraction number sense, with a particular focus on using reasoning and sense-making to compare fractions (rather than procedures that they likely already know and are proficient at using [e.g., using common denominators; cross multiplication; converting to percents or decimals]). Fraction number sense lays the groundwork for further work with fractions such as fraction operations, and is therefore an essential component of the mathematics curriculum for prospective elementary teachers. The key to developing fraction number sense is to have a conceptual understanding of fractions as numbers. Therefore, this sequence of tasks is intended to help prospective elementary teachers move beyond their current (mainly procedural) understandings to develop deep conceptual understandings.

Since our audience is prospective elementary teachers (PTs), we also include tasks that connect to the practice of teaching (e.g., analyzing children's thinking; posing mathematical problems). As such, these tasks support the development of aspects of PTs' mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), including specialized content knowledge and pedagogical content knowledge.

OVERVIEW OF THE TASKS

The chart below lists the specific tasks and the amount of class time needed for each task. Depending on your context, learning goals, and the amount of class time you have, you might choose to do a subset of the tasks.

Task	Amount of class time needed	Materials needed
1. Comparing fractions	120-180 minutes	• Copy of task for each PT
2. Observing a child compare fractions	30-45 minutes	• Copy of task for each PT • *Integrating Mathematics and Pedagogy* (IMAP) video clip[1] (clip #11, minutes 0:00-3:10)
3. Analyzing a children's task	15-20 minutes (alternatively, this could be completed as homework)	• Copy of task for each PT

[1] This can be purchased at: http://www.pearson.ch/HigherEducation/MathematicsStatistics/PrecalculusPrecollegeMaths/1471/9780131198548/IMAP-CD-ROM-Integrating-Mathematics.aspx

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4. Posing problems to elicit specific strategies	20 minutes (plus homework)	• Copy of task for each PT
5. Ordering fractions	30-45 minutes	• Copy of task for each PT • Set of fraction cards for each PT (or for each pair/small group of PTs)

LEARNING GOALS

In this section, we provide a detailed list of the learning goals for the sequence of tasks. It is important to note that some prior knowledge is assumed – specifically, PTs should have an understanding of the meaning of *numerator* and *denominator* (i.e., the denominator refers to the size of the pieces and the numerator refers to the number of pieces), and PTs need to be able to identify and generate fractions that are equivalent to given fractions.

1. PTs will develop and understand the following sense-making strategies to compare and order fractions that can be found in the Common Core State Standards (CCSS) and Lamon (2012):

a. **Same Size Pieces** (i.e., common denominators) (CCSS 3.NF.3d; 4.NF.2):
If the size of the pieces (i.e., the denominators) is the same, the fraction with the greater number of pieces has the greater value (e.g., 4/12 < 5/12).

b. **Same Number of Pieces** (i.e., common numerators) (CCSS 3.NF.3d; 4.NF.2):
If the number of pieces (i.e., numerators) is the same, the fraction with the larger-size pieces has the larger value (e.g., 7/12 > 7/13).

c. **Benchmark Comparison** (CCSS 3.NF.3d; 4.NF.2):
If the fractions have a different number of pieces and different size pieces, comparing the fractions to a benchmark is often useful. Useful benchmarks include (but are not limited to) whole numbers and unit fractions. Several general types of benchmark scenarios can arise:

i. If a fraction can be directly compared to a benchmark, such as when comparing 17/31 to ½, one can often use properties of the benchmark to justify the comparison, without using another strategy. For example, 17/31 > ½ because 17 is more than half of 31; 25/99 > ¼ because 25 is more than ¼ of 99, or 4 times 25/99 = 100/99, which is more than 1.

ii. If one fraction is larger than a benchmark and one fraction is smaller than that benchmark, the fraction that is larger than the benchmark has the larger value. For example, 5/11 < 7/12 because 5/11 < ½ and 7/12 > ½ (i.e., 5/11 < ½ < 7/12).

iii. If both fractions are less than or greater than a benchmark, then the distance each fraction is from the benchmark must be compared:

- If both fractions are less than a benchmark, the fraction that is a shorter distance from the benchmark has the larger value. For example, both 11/12 and 7/8 are less than 1, but they are not the same distance from 1: 7/8 is 1/8 less than 1 and 11/12 is 1/12 less than 1. Since 1/12 < 1/8, 11/12 is closer to 1 than ⅞, so 11/12 > ⅞.
- If both fractions are greater than a benchmark, the fraction that is a larger distance from the benchmark has the larger value. For example, both 5/8 and 11/20 are more than ½, but they are not the same distance from 1/2: 5/8 is 1/8 more than ½ and 11/20 is 1/20 more than 1/2. Since 1/8 < 1/20, 5/8 is a further

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distance from ½ than 11/20 is, so 5/8 > 11/20.

In addition, PTs will develop and understand the following strategy:

d. **Greater Number of Larger Pieces**:
When comparing a greater number of larger-size pieces to a smaller number of smaller-size pieces, the larger fraction is the one that has the greater number of larger-size pieces (e.g., 18/25 > 16/27, because 18 is more than 16 (more pieces) and 25ths are larger than 27ths (larger pieces). It is important to note that this strategy is not appropriate when comparing a smaller amount of larger pieces to a larger amount of smaller pieces (e.g., 2/7 and 3/8) – there is no way to know which fraction is larger without using some other strategy.

2. PTs will develop fraction number sense, including:
   a. Having a general sense of the size of a fraction – e.g., knowing that 9/10 is more than ½ but less than 1, that 9/10 closer to 1 than it is to ½ (i.e., 9/10 is 1/10 less than 1 but 4/10 more than ½); knowing that 3/16 is close to, but less than, ¼.
   b. Identifying which comparison strategies would be useful or appropriate (including selecting an appropriate benchmark, if applicable) to use when comparing or ordering fractions – e.g., when comparing 3/7 and 6/11, recognizing that both numbers are close to ½ so comparing to ½ would be useful; recognizing that the Same Number of Pieces strategy would also be useful since the number of pieces in 6/11 is a multiple of the number of pieces in 3/7; recognizing that the Greater Number of Larger Pieces strategy cannot be used for this comparison because while sevenths are larger than elevenths, there is not a greater number of sevenths than elevenths.

3. PTs will be able to distinguish between valid mathematical arguments and invalid or incomplete mathematical arguments, and be able to construct valid mathematical arguments for comparing and ordering fractions.

4. PTs will analyze children's thinking, including common (mis)conceptions such as:
   a. thinking that 7/8 = 8/9 because both fractions are 1 piece away from one OR the numerators and denominators of the two fractions are both 1 away from each other (numerators 8-7 = 1 and denominators 9-8 = 1)
   b. thinking that 3/6 > ½ because one fraction has a larger numerator and a larger denominator so that fraction must be larger (3 > 1 and 6 > 2)
   c. thinking that 3/6 < ½ because ½ is only 1 piece less than 1 and 3/6 is 3 pieces less than 1
   d. thinking that 4/3 < 1 because all fractions are less than 1

5. PTs will pose fraction comparison problems that elicit the sense-making strategies that they have developed.

FACILITATING THE TASKS

Task 1: Comparing fractions

1. To help PTs lay the groundwork for developing sense-making strategies, begin by having them work individually for 5-10 minutes to explore the following prompts:

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- Make a list of everything you know about the number ⅞.
- Keeping the denominator the same, find 3 fractions that are greater than ⅞, and find 3 fractions that are less than ⅞.
- Keeping the numerator the same, find 3 fractions that are greater than ⅞, and find 3 fractions that are less than ⅞.

Orchestrate a 15-30 minute whole-class discussion in which PTs share and discuss their ideas. In our experience, many PTs comment that ⅞ means 7 out of 8 – while this is one interpretation, it doesn't connect to the idea that ⅞ is a *number* of a particular size, nor does it translate to fractions greater than 1. As PTs make their individual lists about ⅞, look for ideas such as the following and make sure that these emerge during this discussion:

- ⅞ < 1 (Possible reasons include: because the numerator is less than the denominator; 1 = 8/8 and 8/8 is more than ⅞.)
- ⅞ is between 0 and 1 on a number line
- ⅞ is ⅛ less than 1
- ⅞ > ½ (Possible reasons include because 7 is more than half of 8; because ½ = 4/8, and ⅞ is more than 4/8.)
- ⅞ is ⅜ more than ½
- 7/8 = 7 1/8's = ⅛ + ⅛ + ⅛ + ⅛ + ⅛ + ⅛ + ⅛

Have PTs share an example of a fraction whose denominator is 8 that is greater than/less than ⅞ – for example, 9/8 > 7/8. Press PTs to provide explanations for how they know this that are grounded in reasoning and sense-making and that connect to the strategies they will be developing, such as:

  - 9/8 > 1 (b/c 9/8 = 1 + ⅛; or b/c 1 = 8/8 and 9/8 > 8/8) and ⅞ < 1 [for reasons described above] (comparing to a benchmark of 1)
  - Both fractions have eighth-size pieces – 9 eighths > 7 eighths (Same Size Pieces)

Have PTs share an example of a fraction whose numerator is 7, but is greater than or less than ⅞–for example, 7/14 <⅞. Some sense-making explanations include:

  - 7/8 > 7/14 because ⅞ > ½ [for reasons described above] and 7/14 = ½ (Same Size Pieces & comparing to a benchmark of 1/2)
  - 7/14 and ⅞ have seven pieces, but a whole split into 8 equal pieces will have larger pieces than a (same-sized) whole split into 14 equal pieces (i.e., eighths are larger than fourteenths). So ⅞ > 7/14 because 7 larger pieces is greater than 7 smaller pieces. (Same Number of Pieces)

Note: Depending on how much time you have to complete the task activities and how much prior experience your students have had with fractions, you may want to go into more or less detail in this part of the activity. We have found it important to model complete explanations during this launch, so that PTs understand what we mean by "provide a sense-making explanation" in the written part of the activity – for example, concluding that 7/9 < ⅞ because "7/9 is farther away from 1" is not an acceptable explanation because it lacks justification for why 7/9 is further from 1 or why being further from 1 means that 7/9 is smaller.

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2. Distribute the 15 problems to PTs and have them work individually for 5-10 minutes, and then continue to work on the task with a partner or in a small group for 20-30 minutes. (If you are concerned that PTs will rely on the common denominator strategy, you may want to specify that they can only use the common denominator strategy if they can do so by renaming <u>only one</u> of the fractions, or encourage PTs to try to solve the problems using different strategies.)

3. Monitor PTs in their individual and partner/small group work (making a note of those who develop the sense-making strategies so that they can present their work during the whole-class discussion), and assist them as needed. For example:
   - In our experience, PTs sometimes need to be reminded that their justifications shouldn't rely solely on a picture. If PTs seem to be relying on pictures, encourage them to look across the entire set of 15 problems and consider whether they could determine the greater fraction for any of the problems without a picture (e.g., PTs often feel that #7 (15/17 vs. 19/18) is one of the easier problems since the fractions can be easily compared to 1). Encourage PTs to consider whether their thinking could apply to some of the other problems. You may also want to pull the class together after a few minutes and have a PT share a sense-making strategy to one of the problems, in order to establish justification routines for the class.
   - If PTs are struggling with particular problems, ask them how they solved problems that we know are similar and whether they could apply a similar strategy (Appendix A lists the problems and the strategies that can be applied to each problem). For example, if PTs are struggling with #6 (13/15 vs. 17/19), you might ask them how they solved #5 (8/9 vs. 12/13) and if their thinking could be used to solve #6.
   - Watch for a common misconception – thinking that fractions that are missing the same number of pieces (e.g., 8/9 and 12/13) are equivalent. If PTs conclude that the fractions in #5 and/or #6 are equivalent, you might want to let it go for the moment and revisit it during the whole-class discussion. Alternatively, you might ask PTs to compare two fractions such that it is more obvious that this thinking is not valid, such as ¾ and 99/100.
   - If some groups finish early, you could extend their work as follows:
     - Ask them to look across the set of 15 problems and consider whether any of the problems could be solved in more than one way (e.g., #4 and #11 can be solved in multiple ways).
     - Ask them to look across the set of 15 problems and consider whether any of the strategies they developed could be applied to any other problems.

4. Orchestrate a whole-class discussion in which PTs share and discuss the strategies they developed. Consider the following suggestions:
   - This discussion will likely take at least 1 hour, and the focus should be on developing and explaining the strategies – it is not necessary to discuss all 15 problems. You might want to organize the discussion such that the following problems and strategies are presented in the order shown below:
     - #3 (Same Size Pieces)
     - #2 (Same Number of Pieces)
     - #1 (Benchmark; one fraction is the benchmark and one fraction is greater or less than the benchmark)
     - #7 (Benchmark; one fraction greater and one fraction less than the benchmark)

- #5 (Benchmark; both fractions less than the benchmark)
- #15 (Greater Number of Larger Pieces)

- After each strategy is discussed, give it a name so that PTs can refer to it later (once all the strategies are discussed and named, we have found it helpful to provide PTs with a list of the strategies (shown in Figure 1) and their generalizations so they can refer to it as needed).

Strategy 1: Same-size pieces (e.g., #3)
- When comparing same-size parts, the fraction with the greater number of parts being considered has the greater value.

Strategy 2: Same number of pieces (e.g., #2)
- When comparing fractions in which the same number of parts are being considered, but the parts are of different sizes, the fraction with the larger sized parts has the greater value.

Strategy 3: Comparing to a benchmark fraction
- Helpful when comparing fractions with different sized parts and different number of parts.
- If one fraction is larger than our benchmark, and one fraction is smaller than our benchmark, the fraction that is larger than our benchmark is greater than the fraction that is smaller than our benchmark (e.g., #7).
- If both fractions are larger (or smaller) than our benchmark, then we need to consider how much more (or less) each fraction is than our benchmark in order to determine the greater fraction and compare those amounts using one of our other strategies in order to determine the greater fraction (e.g., #5).

Strategy 4: Greater number of larger pieces (or, lesser number of smaller pieces) (e.g., #15)
- If we have *more* pieces that are *larger-sized*, and we're comparing that to having *fewer* pieces that are *smaller-sized*, the fraction that has more pieces that are larger-sized has the greater value.
- But, if we have *fewer* pieces that are *larger-sized*, and we're comparing that to having *more* pieces that are *smaller-sized*, then we cannot determine which fraction has the greatest value (without using some other strategy).

Figure 1. A list of the strategies to share with PTs.

- After each strategy is discussed and named, you might also want to ask PTs to look across the set of 15 problems and identify other problems that could be solved using that strategy. (In our experience, PTs often spontaneously comment about this).
- If none of the PTs used a particular strategy that you want them to develop (in our experience, the Greater Number of Larger Pieces strategy does not always emerge naturally), you might present a solution that uses the strategy and ask PTs if they think the strategy is valid and to explain why or why not. Once PTs come to agreement that the strategy is valid, you might ask PTs if the strategy could be applied to any of the 15 problems. Alternatively, you could present them with the problem situation shown in Appendix B in order to give PTs an opportunity to develop the strategy themselves.
- Ask PTs how they decided what benchmark to use on the benchmark problems. They will likely respond that a good benchmark is one that is "close to" both fractions. Flag this as a rule of thumb that may be useful.
- Discuss any misconceptions that emerged from PTs' work (e.g., thinking that fractions that are missing the same number of pieces [e.g., 8/9 and 12/13] are equivalent).

Task 2: Observing a child compare fractions

1. Begin by distributing the Task 2 handout to PTs and have them consider Questions 1 & 2 individually for a few minutes. Ask PTs to share their thinking on Question 1, making sure that they provide sense-making arguments for how they know which fraction is greater. Then ask PTs to share possible explanations for incorrect answers, recording their ideas on the board so that the class can refer back to them after watching the video clip. In our experience, PTs usually anticipate that students may think that all fractions are less than one (which would lead to incorrectly stating that 1 > 4/3); students may think that the fraction with the larger numerator and/or denominator is the greater fraction (as in the case of 3/6 vs ½); or students may think that ½ > 3/6 because 3/6 is three pieces less than 1, but ½ is only 1 piece less than 1. We have also found that PTs often struggle to think of a reason why students would incorrectly conclude that the 1/7 > 2/7.

2. Show the 3-minute video clip of an interview with Ally, an average fifth-grade student from a high-performing school (San Diego State University Foundation, Philipp, & Cabral, 2004), who is asked to solve a set of fraction comparison problems, including the three problems in Question 1. (Alternatively, in order to save class time, you could provide PTs with a handout that includes Ally's written work, annotated with a few sentences that capture what Ally says about each comparison problem, so that PTs can engage in the activity as homework.) Ask PTs to keep the following question in mind as they watch the clip: *What are Ally's conceptions?*

   In our experience, PTs often have difficulty making sense of Ally's thinking the first time they view the clip. We have found it helpful to play the clip, then to have PTs have a brief discussion in their small groups about Ally's thinking, and then to play the clip again, this time stopping after each problem that Ally explains to discuss it as a whole class. Consider the following suggestions for the whole-class discussion:

   - After each problem, have PTs describe Ally's thinking and the misconceptions that she appears to have (e.g., the fraction whose denominator is closest to 1 is the larger fraction; all fractions are less than 1).
   - You might also ask PTs to consider what other questions they would want to ask Ally (if they could interview her) and why. For example, PTs often suggest asking Ally to compare fractions such as 1 and 4/4, in order to investigate her misconception that all fractions are less than 1.

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Task 3: Analyzing a Children's Task

This task can be initiated after completing Task 1 or Tasks 1 and 2. Distribute this task for PTs to have an opportunity to analyze the types of fraction comparison strategies that an authentic elementary grade lesson might address. We generally use this task as a homework assignment, but it could be done during class, if time permits. Table 1 lists the problems and what we consider to be the target strategies.

Table 1. Problems in the children's task and target strategies.

Problem	*Fractions being compared (underlined fraction is the greater fraction)*		*Strategy or strategies that can be used to compare the fractions*
1.	7/10	3/5	Use equivalent fractions to make Same Size Pieces (3/5 = 6/10 < 7/10)
2.	7/8	9/10	Compare both fractions to a Benchmark of 1 and then compare the distance each fraction is from 1 using Same Number of Pieces (e.g., ⅞ is ⅛ less than 1 and 9/10 is 1/10 less than 1. ⅛ > 1/10 [Same Number of Pieces], so ⅞ is missing a larger amount than 9/10 is, which means that 9/10 > ⅞)
3.	4/3	3/4	Compare both fractions to a Benchmark of 1 (4/3 > 1 because 3/3 = 1, and ¾ < 1 because 4/4 = 1, so 4/3 > ¾)
4.	3/8	1/3	Use equivalent fractions to make Same Number of Pieces (1/3 = 3/9 < 3/8)

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Task 4: Posing Problems to Elicit Specific Strategies

Task 4 asks PTs to design some fraction comparison problems that would elicit the sense-making strategies developed by the class. There are two versions of Task 4 – one of which has PTs solve each other's problems and one of which does not – choose the version that fits your needs.

1. Before assigning Task 4 for homework, we have found it helpful to engage PTs in a brief discussion so that there is consensus about what makes a good fraction comparison problem (e.g., a good problem is challenging but solvable with the sense-making strategies developed by the class). Begin by asking PTs to spend 5 minutes creating some fraction comparison problems that could be solved using benchmark strategies. As PTs write problems, monitor their work and make a note of a few problems (or have some problems prepared in advance to share). Record the problems on the board and ask PTs to consider how they would solve each problem. Then orchestrate a brief whole-class discussion (15-20 minutes) in which PTs share how they solved each problem and then compare to the strategy being targeted. For example, Table 2 presents problems in which the author intended to elicit the *compare to ½ benchmark strategy* and commentary on the quality of each problem:

Table 2. Examples of problems posed by PTs.

Fraction comparison problem	*Strategy the author was targeting*	*Comments on the quality of the problem*
5/12 or 9/17	Benchmark of ½ (5/12 < ½; 9/17 > ½)	Comparing to ½ is an ideal strategy to use for this problem – since 5/12 and 9/17 contain "messy" numbers, it is not easy to rename one of the fractions so that Same Size Pieces or Same Number of Pieces can be used; nor does Greater Number of Larger Pieces apply. Also, it is not easy to tell at first glance which fraction is larger, so a sense-making explanation is encouraged.
1/16 or 11/12	Benchmark of ½	While a benchmark of ½ *could* be used to solve this problem, it is rather obvious that 1/16 is much smaller than 11/12. This problem does not require much thought on the part of the solver, so it would not help develop number sense.
2/4 or 5/4	Benchmark of ½	This problem elicits a variety of strategies (for example: Same Size Pieces; renaming 2/4 as 5/10 and then using Same Number of Pieces; comparing to a benchmark of 1), so it will not likely elicit the author's intended strategy.

2. Assign Task 4 as homework. If time permits, you might want to have PTs solve each other's problems and discuss them in class. However, in our experience, we have found that it is best to examine their problems in advance and select a subset of problems for them to solve, rather than having PTs exchange problems with a partner (since some PTs may not complete the homework assignment and some PTs' problems may not be as challenging as others).

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Task 5: Ordering Fractions

1. Begin by having PTs work with a partner to order the fractions from smallest to greatest (you might ask PTs to print a set of cards, cut them out, and bring them to class so that you don't have to create a set of cards for each pair). Be sure to press PTs to write their rationale(s) for their arrangements and to specify which strategies they used and for which cards. Alternatively, to save class time, you could have PTs work on this task individually as homework, and begin class with the whole-class discussion (see #3 below).

2. Monitor PTs in their partner work, and assist them as needed. For example:
   - If PTs are have difficulty getting started, suggest that they spread out the cards on their desk or table and begin by ordering any fractions that stand out to them as easy or easier to order (e.g., whole numbers), and then revisit any cards that are more difficult to order later. Alternatively, you might suggest that they make a number line and label some benchmarks (e.g., ¼, ½, ¾, 1, 1½, 2) and then put the appropriate cards into the intervals between these benchmarks.
   - If some groups finish early, you might ask them to consider the following extension:
     - Add some more cards that would pose a challenge for the PTs in our class. What do your new cards prompt teachers to think about? What strategies are you trying to elicit?

3. Orchestrate a whole-class discussion in which PTs come to an agreement on the order of the fractions and share and discuss the strategies they used. Consider the following suggestions:
   - You might begin the discussion by asking a pair of PTs to select 2-3 of the teacher cards (which are large enough for the whole class to see), post them on the board, and explain their thinking. Then have other pairs add to the ordering on the board by posting additional cards and provide justification for their ordering.
   - Ask PTs if anyone included cards that were not in the set – e.g., in our experience, PTs often add the fraction 1½ to their ordering to help them compare 23/16 and 37/24.
   - If PTs do not spontaneously comment that some of the cards can be ordered using multiple justifications, be sure to ask them to consider this – e.g., in our experience, PTs have compared 3/16 and 5/24 both by comparing to ¼ and by comparing to 1/5.

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